Exam Symmetry in Physics

Date	February 2, 2012
Room	5419.0113
Time	9:00 - 12:00
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- You are not allowed to use the lecture notes, nor other notes or books
- All subquestions (a, b, etc) of the three exercises have equal weight
- Answers may be given in Dutch
- Illegible handwriting will be graded as incorrect
- Good luck!

Exercise 1

Consider a regular pentagon and its symmetry group, the dihedral group D_5: $\mathrm{gp}\{c, b\}$ with $c^5 = b^2 = (bc)^2 = e$.

(a) Argue, using geometrical arguments, that D_5 has four conjugacy classes:

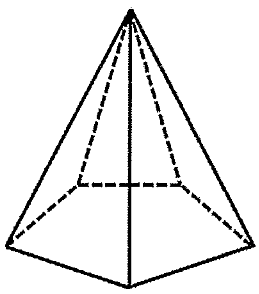
$$(e), (c), (c^2), (b)$$

(b) Determine the dimensions of all inequivalent irreps of D_5.

(c) Construct the character table of D_5, using for instance the variables $x \equiv \cos(2\pi/5) = -\frac{1}{4}(1 - \sqrt{5})$ and $y \equiv \cos(4\pi/5) = -\frac{1}{4}(1 + \sqrt{5})$, that satisfy $x^2 + y^2 = \frac{3}{4}$ and $xy = -\frac{1}{4}$.

(d) Identify the proper invariant subgroup(s) of D_5.

Consider a regular five-sided pyramid with a regular pentagon as base (see figure) and its symmetry group C_{5v}.



(e) Show that $C_{5v} \cong D_5$ using cycle notation.

(f) Construct explicitly the three-dimensional vector representation D^V for the generators of both C_{5v} and D_5, where D_5 is viewed as a subgroup of the rotations in three dimensions.

(g) Decompose D^V of both D_5 and C_{5v} into irreps and use this to conclude whether these groups allow in principle for an invariant three-dimensional vector, such as an electric dipole moment.

Exercise 2

Consider a classical potential $V(|\vec{r}|)$ experienced by a charged particle at position $\vec{r} = (x, y, z)$ in three dimensions.

(a) Show that the potential V is invariant under $O(3)$ transformations.

(b) What is the defining representation of $O(3)$?

(c) Describe under which representations of $O(3)$ the force $\vec{F} = -\vec{\nabla}V$ and the torque $\vec{\tau} = \vec{r} \times \vec{F}$ transform.

(d) Use symmetry arguments to argue that the torque must vanish for the given potential V.

Next consider the case of a uniform electric field. A particle with charge q will then experience a Lorentz force $\vec{F} = q\vec{E}$.

e) What is the group of isometries (transformations that preserve distances) under which the corresponding potential is invariant?

f) Use symmetry arguments to argue that there cannot be a nonzero torque either in this case.

Exercise 3
Consider the angular momentum operators L_i $(i = 1, 2, 3)$ acting on the states $|l, m\rangle$.

(a) Explain why the eigenvalues of the operator $\vec{L}^2 = \sum_i L_i^2$ label the irreducible representations of a Lie group with Lie algebra $[L_i, L_j] = i \sum_k \epsilon_{ijk} L_k$.

(b) Write down the explicit matrix for L_z acting on the space of $|l, m\rangle$ states for non-negative integer l.

(c) Use the result of (b) to show that the characters of $SO(3)$ matrices are of the form:

$$\chi^{(l)}(\theta) = 1 + 2\left(\cos(\theta) + \ldots + \cos(l\theta)\right)$$

(d) Draw a picture of the parameter space of $SO(3)$ and indicate the conjugacy classes.

(e) Explain the relation between the $l = 1$ irrep and the vector representation of $SO(3)$.